Dynamics of Gravity-Oriented Orbiting Systems with Application to Passive Stabilization

B Etkin*
Univei sity of Toronto, Toronto, Ontaı io, Canada

A theoretical framework is presented for analyzing the motion of a multibody satellite system in a gıavity-oriented orbiting reference frame It consists essentially of expressions for the forces and moments of the body-force field on arbitrary bodies and of their utilization in Lagrange's equation to find the equations of motion It is then applied to the analysis of a specific system designed for passive attitude stabilization The equations are linearized and separated into two groups (longitudinal and lateral), and numerical solutions are obtained Damping to half-amplitude in times of the order of 0 3 orbit is possible in either system separately For combined three-axis stabilization, somewhat longer times would result from inevitable compromises The effect of solar radiation on the system is briefly discussed

Nomenclature

A_1, A_2, A_3	= constants [Eqs (25)]
a	= distance from satellite mass center to stabilizer hinge
b	= half-length of one pitch roll stabilizer rod
b'	= half-length of one yaw stabilizer rod
(A, B, C)	= principal centroidal moments of inertia of a body
(A', B')	= pıincipal centroidal moments of inertia of one pitch-roll stabilizer
B_1, B_2, B_3	= constants [Eqs (23)]
c_1	= pitching resistance coefficient of hinges
c_2	= rolling resistance coefficient of hinges
d	= differential operator $d/d\gamma$
e	= eccentricity of ellipse
$\mathfrak{F}$	= generalized force in Lagrange's equation
$\mathbf{f}_g$	= gravitational force per unit mass
$\mathbf{f}_i$	= inertia force per unit mass
$\mathbf{F}(F_x, F_y, F_z)$	= total force acting on a body
$\mathbf{G}(L, M, N)$	= total moment about the mass center of a body
g_0	= μ/R^2
I	= added moment of inertia of roll-yaw coupler
$\hat{I}$	= I/A_b
I_η	= moment of inertia [Eq (10)]
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	= unit vectors along x, y, z
$J_{\xi\eta}$, etc	= products of inertia [Eq (10)]
m	= total mass of satellite
m'	= mass of one pitch-roll stabilizer rod
q_k	= generalized coordinate
$\mathbf{r}$	= position vector of particle (Fig 1)
$\mathbf{R}$	= position vector of origin (Fig 1)
R_0	= average value of R
R_p	= perigee distance
t	= time
T	= kinetic energy, or period of orbit
W	= work done in a virtual displacement
$Oxyz$	= gravity oriented reference frame (Fig 1)
$\alpha, \beta, \kappa\ \lambda$	= angular displacements of stabilizer rods (Fig 3)
μ	= gravitational constant
ω	= angular velocity $\dot{\gamma}$
ω_0	= average value of ω
$\boldsymbol{\rho}$	= position vector of particle (Fig 2)
$\xi, \eta\ \zeta$	= centroidal coordinate axes for body (Fig 2)
$\psi, \theta\ \phi$	= Euler angles, defining rotation of body relative to $C\xi\eta\zeta$
γ	= polar angle (true anomaly) (Fig 1)
Γ_i	= c_i/ω_0, $i = 1, 2$
$\hat{\Gamma}_1$	= Γ_1/B_b
$\hat{\Gamma}_2$	= Γ_2/A_b
$(\ )_b$	= satellite body

Received August 26 1963; revision received March 25, 1964 This work was supported by the U S Air Force Office of Scientific Research, Grants No 62 40 and 222-63 The valuable assistance of H Maeda of Kyoto University, while on leave at Toronto, is gratefully acknowledged

* Professor of Aerospace Engineering, Institute for Aerospace Studies Associate Fellow Member AIAA

I Introduction

THIS paper deals with the dynamical theory of a ceıtain class of satellites that are composed of more than one body, coupled by frictional or elastic elements The class consists of those satellites which, in their designed mode of opeıation, maintain one axis fixed (or nearly fixed) along the local gravity vector It includes satellites designed for meteorological, geophysical, communication, and reconnaissance missions, for all of which the local vertical is the "natural" reference direction

Although the analysis was motivated by the problem of passive stabilization of earth satellites by means of the well-known gravity-gradient principle, and Sec III is devoted to it, the general equations given in Sec II have wider application They can, in fact, be applied to the formulation of the configuration equations of motion for virtually any system, rigid or elastic

It seems natural to choose a frame of reference for the equations of motion in which the system, in its intended state, either is at rest or has at most some simple motion If one axis of the vehicle is to be vertical, then the orbiting reference frame $Oxyz$ shown in Fig 1 meets this requirement If the orbit is a circle, the vehicle is at relative rest; if it is elliptic, the vehicle has a steady pitching oscillation † Since this reference frame is non-Newtonian, having both rotation and linear acceleration of the origin, the external force field must be modified so as to include the appropriate "inertia forces "

The origin O of the coordinate system is at the mass center $(\mathbf{R}, \gamma)$ of the orbiting system It can easily be shown that to the first order [i e , when Eqs (2) below are valid] there is no back-coupling of the rotational and distortional motions of the vehicle on the tıajectory of the mass center The latter therefore describes a Keplerian orbit, and the motion of the reference frame is known a priori We neglect oblatness of the earth and the presence of other astronomical bodies; the orbit is therefore a planar ellipse The burden of what follows may be briefly summarized thusly:

1) The gravitational field is expressed as a linear function of distance in the immediate neighborhood of the vehicle To it is added the inertia force field associated with the motion of the reference frame to give the total effective body-force field This is integrated to give the total force and

† Simple harmonic if $e \ll 1$

moment acting on a body of arbitrary shape and location within the orbiting frame

2) The derived expressions for force and moment are then used, by means of Lagrange's equation, to obtain the equations of motion of a particular multibody satellite (designed as a passive stabilization system)

3) Numerical solutions are given for the small-disturbance characteristic modes and for the forced oscillation caused by orbit ellipticity

4) The influence of solar radiation on the system is discussed briefly

II Basic Equations

The following analysis is given in abbreviated form A complete account, with all the details, is contained in Ref 4

Body-Force Field

The gravitational force per unit mass is (see Fig 1)

$$\mathbf{f}_g = -(\mu/r^3)\mathbf{r} \tag{1}$$

Since $(x/R, y/R, z/R)$ are of order 10^{-6}, it is a very good approximation to linearize Eq (1) Its scalar components then become

$$\left.\begin{aligned} f_{gx} &= -g_0(x/R) \\ f_{gy} &= -g_0(y/R) \\ f_{gz} &= g_0[1 + 2(z/R)] \end{aligned}\right\} \tag{2}$$

where $g_0 = \mu/R^2$ is the value of gravity at the origin

The angular velocity of the reference frame is $-\mathbf{j}\omega$, and the acceleration of its origin is

$$\mathbf{a}_0 = \mathbf{k}g_0 \tag{3}$$

where $g_0 = R\omega^2 - \ddot{R}$ It follows that the inertia force field per unit mass due to motion of the reference frame is given by[2]

$$\left.\begin{aligned} f_{ix} &= \omega^2 x + \omega z + 2\omega\dot{z} \\ f_{iy} &= 0 \\ f_i &= -g_0 + \omega^2 z - \omega x - 2\omega\dot{x} \end{aligned}\right\} \tag{4}$$

These relations, which are linear in the coordinates, are, unlike Eqs (2), exact It is convenient to transform fiom time as the independent variable to the polar angle γ by the relation

$$d/dt = \omega(d/d\gamma)$$

When Eqs (4) have been so transformed and added to Eqs (2), we obtain the total force field:

$$\left.\begin{aligned} f_x &= x\left(\omega^2 - \frac{g_0}{R}\right) + \frac{1}{2}z\frac{d\omega^2}{d\gamma} + 2\omega^2\frac{dz}{d\gamma} \\ f_y &= -y\frac{g_0}{R} \\ f_z &= z\left(\omega^2 + 2\frac{g_0}{R}\right) - \frac{1}{2}x\frac{d\omega^2}{d\gamma} - 2\omega^2\frac{dx}{d\gamma} \end{aligned}\right\} \tag{5}$$

The motion of a free particle in this force field has recently been discussed by Knollman and Pyron[6]

Force and Moment on a Finite Body

The total force and moment acting on a finite body (see Fig 2) are

$$\left.\begin{aligned} \mathbf{F} &= \int \mathbf{f}\, dm \\ \mathbf{G} &= \int \boldsymbol{\rho} \times \mathbf{f}\, dm \end{aligned}\right\} \tag{6}$$

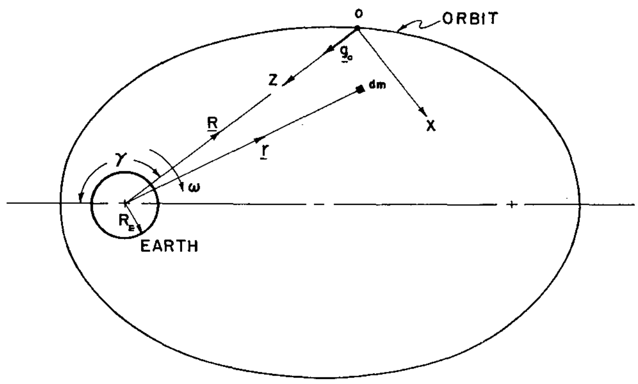


Fig 1 Orbiting ıefeıence frame

where $\boldsymbol{\rho}$ is the position vector of dm relative to the body center $C(\bar{x}, \bar{y}, \bar{z})$, and hence $\mathbf{G}$ is the moment about the mass center

Integrating Eqs (5) for the resultant force, we get

$$\left.\begin{aligned} F_x &= m[a_1\bar{x} + a_2\bar{z} + a_3(d\bar{z}/d\gamma)] \\ F_y &= mb_1\bar{y} \\ F &= m[c_1\bar{x} + c_2\bar{z} + c_3(d\bar{x}/d\gamma)] \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} &a_1 = \omega^2 - \frac{g_0}{R} & & & &c_1 = -\frac{1}{2}\frac{d\omega^2}{d\gamma} \\ &a_2 = \frac{1}{2}\frac{d\omega^2}{d\gamma} & &b_1 = -\frac{g_0}{R} & &c_2 = \omega^2 + 2\frac{g_0}{R} \\ &a_3 = 2\omega^2 & & & &c_3 = -2\omega^2 \end{aligned}\right\} \tag{8}$$

The integral for the moment yields

$$\left.\begin{aligned} L &= \bar{a}_1 J_{\xi\eta} + \bar{a}_2 J_{\eta\zeta} + \bar{a}_3 \int \eta(d\xi/d\gamma)dm \\ M &= \bar{b}_1 J_{\xi\zeta} + \bar{b}_2 I_\eta + \bar{b}_3(dI_\eta/d\gamma) \\ N &= \bar{c}_1 J_{\xi\eta} + \bar{c}_2 J_{\eta\zeta} + \bar{c}_3 \int \eta(d\zeta/d\gamma)dm) \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} &\bar{a}_1 = -\frac{1}{2}\frac{d\omega^2}{d\gamma} & &\bar{b}_1 = -3\frac{g_0}{R} & &\bar{c}_1 = -\omega^2 \\ &\bar{a}_2 = \omega^2 + 3\frac{g_0}{R} & &\bar{b}_2 = \frac{1}{2}\frac{d\omega^2}{d\gamma} & &\bar{c}_2 = -\frac{1}{2}\frac{d\omega^2}{d\gamma} \\ &\bar{a}_3 = -2\omega^2 & &\bar{b}_3 = \omega^2 & &\bar{c}_3 = -2\omega^2 \end{aligned}\right\} \tag{10}$$

and $J_{\xi\eta} = \int \xi\eta dm$, etc , $I_\eta = \int (\xi^2 + \zeta^2)dm$

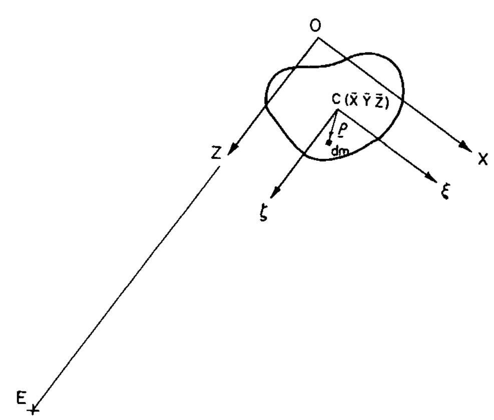


Fig 2 Body centered coordinate system $C(\bar{x}, \bar{y}, \bar{z})$ is body mass center, and $C\xi\eta\zeta$ are parallel to $Oxyz$

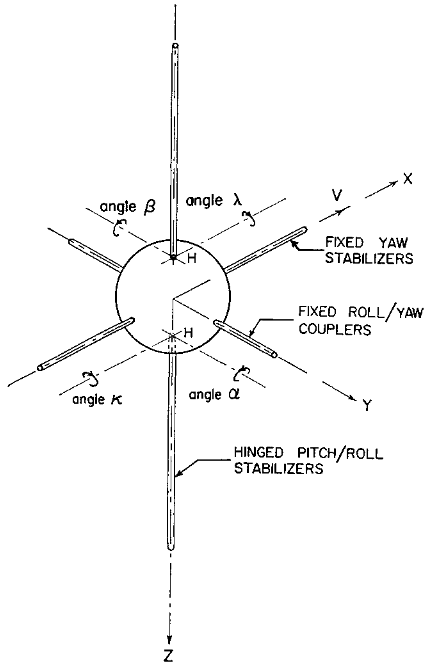


Fig 3 Diagram of two-hinged stabilizer system

In the preceding equations for the force and moment, the coefficients are all functions of γ The explicit relations that express the dependence for an elliptic orbit defined by (R_p,e) are readily derived from basic orbital theory (see, e g , Ref 3) as

$$\omega^2(\gamma) = [\mu/R_p^3(1+e)^3](1+e\cos\gamma)^4 \tag{11}$$

$$R(\gamma) = R_p(1+e)(1+e\cos\gamma)^{-1} \tag{12}$$

$$(g_0/R)(\gamma) = [\mu/R_p^3(1+e)^3](1+e\cos\gamma)^3 \tag{13}$$

When the orbits are of small eccentricity $e \ll 1$, which is a case of interest, the foregoing expressions can be approximated to the first order in e by

$$\left.\begin{aligned} \omega^2(\gamma) &= \omega_0^2(1+4e\cos\gamma) \\ R(\gamma) &= R_0(1-e\cos\gamma) \\ (g_0/R^2)(\gamma) &= \omega_0^2(1+3e\cos\gamma) \end{aligned}\right\} \tag{14}$$

all of which give simple harmonic variations with the independent variable γ; the mean values of $R(\gamma)$, $\omega^2(\gamma)$ are given by

$$R_0 = R_p(1+e)$$

$$\omega_0^2 = \mu/R_p^3(1+3e) = \mu/R_0^3$$

By means of Eqs (14), first-order approximations for the a_i c_i, Eqs (8) and (10), are readily written down (They are given in full in Ref 4)

The moment expressions, Eqs (9), are seen to contain the moment of inertia in pitch, $I\eta$, and the three products of

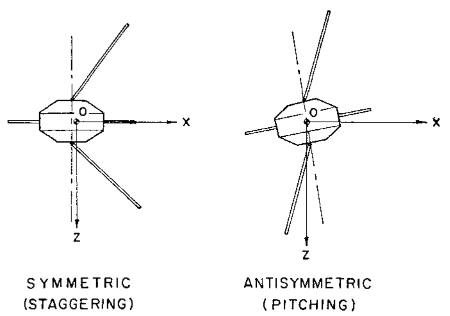


Fig 4 Characteristic longitudinal modes

inertia of the body with respect to the gravity-oriented centroidal axes $C\xi\eta\zeta$ Since the body in general rotates with respect to these axes and may in addition experience nontrivial elastic or thermal deformation, the inertia coefficients in general vary with time, i e , they are functions of γ In application, it is convenient to express them in terms of the principal moments of inertia, A, B, C, which are constant if the body is rigid Let the principal axes be oriented relative to $C\xi\eta\zeta$ by the three successive rotations ψ, θ, ϕ, following the conventional practice in flight dynamics The required inertia coefficients are obtained by the transformation of the inertia tensor[5]

$$[I]_{\xi\eta\zeta} = [l]\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}[l]^* \tag{15}$$

where $[I]_{\xi\eta\zeta}$ is the inertia matrix in the $\xi\eta\zeta$ system, and $[l]$, $[l]^*$ are the matrix of direction cosines and its transpose When ψ, θ, ϕ are small angles, the required inertias reduce, to the first order, to

$$I_\eta = B \qquad J_{\xi\eta} = (B-A)\psi$$

$$J_{\eta\zeta} = (C-B)\phi \qquad J_{\zeta\xi} = (A-C)\theta \tag{16}$$

Separation into Two Groups

If Oxz is a plane of symmetry of the system, then a "longitudinal" motion is possible, like that of airplane dynamics, in which each particle moves in a plane parallel to the orbital plane The required forces and moments for any body are then F_x, F , and M of Eqs (7) and (9) For such a motion to take place, there must of course be no external forces (e g , solar radiation) that produce appreciable nonzero values of F_y, L, and N There is no restriction as to amplitude, however, and the pitch angles θ of the bodies that make up the satellite may be large

Again, just as in dynamics of airplanes, we find that "lateral" motion, involving displacements $\bar{y}$, ψ, ϕ of a body, can exist independently of the longitudinal motion when all second- and higher-order terms are negligible The analogy with airplanes is further preserved in that rolling and yawing are, in general, coupled

III Application to a Passive Stabilization System

The equations of Sec II have been applied to the analysis of a particular system proposed[4 7] for passive three-axis attitude stabilization The system is shown in Fig 3 It consists of a satellite body that carries two vertical rods‡ (pitch-roll stabilizers), universally hinged at H so that both pitching and rolling displacements may occur The hinges contain dissipative elements (electrodynamic, hydrodynamic, or other) such that rotation of the rods relative to the body produces resistive couples proportional to rate These couples are the means by which damping of all modes is obtained Depending on the mass distribution of the satellite body, additional arms may be required on one or both of the x and y axes That on the x axis is denoted a "yaw stabilizer"; that on the y axis is called a "roll-yaw coupler" In Ref 7, a more general case is treated, in which the yaw stabilizers are also hinged, so that they are free to swing in yaw No significant technical advantage was found in this arrangement, however, and so the present paper describes only the case of fixed yaw stabilizers

The particular system chosen for study herein is of course only one of an infinite variety of possible ones (a number of others have been described in the literature; see, for example,

‡ The technical development of suitable extensible rods is now an accomplished fact; see Ref 8

Refs 10 and 11§) that can, in principle, be used for passive stabilization No special claims are made for it, other than that it is compact, symmetrical, appears to be quite within the state of the art, and has excellent performance The full details of the analyses are contained in Refs 4 and 7 and are not reproduced here in the interests of brevity Instead, we present only an outline

First the equations are linearized by considering only small disturbances and by assuming that the orbit eccentricity is small Thus $e \ll 1$ and terms containing products of e with the first-order variables, θ, α, β, etc , are assumed negligible This approximation would be invalid if the motion variables, θ, α, etc , were much smaller than e, for then terms such as $e\theta$ would be much larger than terms such as θ^2 and their neglect not justified The solutions would then be invalid in the limit of vanishing motion Now when $e \neq 0$, a parametrically forced oscillation in the longitudinal degrees of freedom persists indefinitely, in which the amplitudes of the motion are generally greater than e Hence the motion never does vanish, and the forementioned restriction is not significant for the longitudinal motion It does come into force for the lateral modes, however, since then there is no fiist-order parametric excitation by orbit eccentricity The lateral solutions given are therefore exact for circular orbits, but they neglect a second-order residual motion caused by e when the orbit is elliptical

The linearized expressions for the forces and moments on a body are found to be as follows:

Longitudinal

$$\left.\begin{aligned} F_x &= m\omega_0^2[(e\cos\gamma)\bar{x} - (2e\sin\gamma)\bar{z} + 2(d\bar{z}/d\gamma)] \\ F_z &= m\omega_0^2[(2e\sin\gamma)\bar{x} + (3 + 10e\cos\gamma)\bar{z} - 2(d\bar{x}/d\gamma)] \\ M &= -\omega_0^2[3(A - C)\theta + 2eB\sin\gamma] \end{aligned}\right\} \quad (17)$$

Lateral

$$\left.\begin{aligned} F_y &= -m\omega_0^2\bar{y}(1 + 3e\cos\gamma) \\ L &= \omega_0^2[4(C - B)\phi + (A + C - B)(d\psi/d\gamma)] \\ N &= -\omega_0^2[(B - A)\psi + (A + C - B)(d\phi/d\gamma)] \end{aligned}\right\} \quad (18)$$

The next step is to obtain the equations of motion of the system Two principal methods of attack are available: 1) to write down Euler's equations of motion foi each constituent body, thus obtaining $6n$ equations for n bodies; this is essentially the approach of Roberson[1]; and 2) to use Lagrange's equation and obtain only as many equations as there are degrees of freedom

When the system consists only of rigid bodies, the first method is always possible It has the advantage of supplying the inertia terms "ready-made " Disadvantages are that it leads to more equations than there are degrees of freedom, the difference being equal to the number of unknown force and moment interactions between the bodies, and that it cannot be applied at all to elastic bodies

The second method, that of Lagrange, is essentially unrestricted; it can be applied in all situations However, it is not always the shortest path to the final equations when an alternative exists Nevertheless, the second method was the one chosen for the present study The kth equation of motion is given by

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = \mathfrak{F}_k \quad (19)$$

where T is the total kinetic energy of the system relative to the orbiting frame of reference and $\mathfrak{F}_k$ is the generalized force

§ Published after this paper was first submitted; references added after review

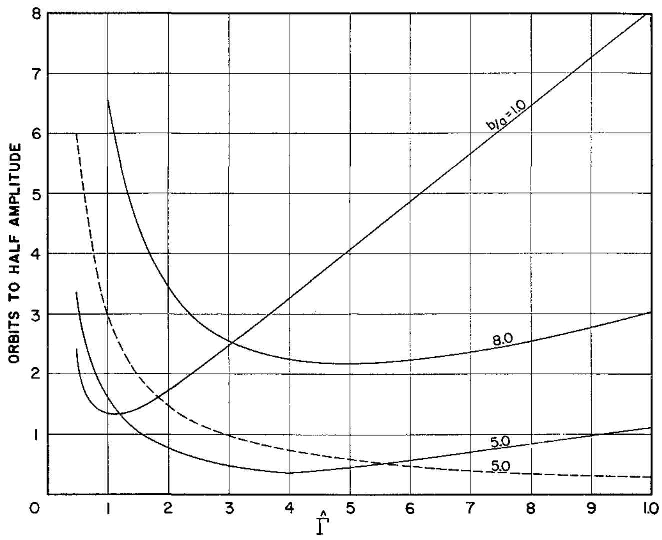


Fig 5 Number of orbits to half-amplitude for least-damped longitudinal mode Solid line indicates antisymmetric modes, and dashed line indicates symmetric mode

The details of the computation of T and its derivatives are given in Refs 4 and 7, with the result that the equations of motion are as follows:

Longitudinal

$$\left.\begin{aligned} B\ddot{\theta} + B_1\ddot{\alpha} + B_1\ddot{\beta} &= \mathfrak{F}_\theta \\ B_1\ddot{\theta} + B_2\ddot{\alpha} + B_3\ddot{\beta} &= \mathfrak{F}_\alpha \\ B_1\ddot{\theta} + B_3\ddot{\alpha} + B_2\ddot{\beta} &= \mathfrak{F}_\beta \end{aligned}\right\} \quad (20)$$

where

$$\left.\begin{aligned} B &= \text{total moment of inertia about the } \eta_b \text{ axis} \\ B_1 &= B' + m'b(a + b) \\ B_2 &= B' + m'b^2[1 - (m'/m)] \\ B_3 &= (m'/m)(m'b^2) \end{aligned}\right\} \quad (21)$$

Lateral

$$\left.\begin{aligned} A\ddot{\phi} + A_1\ddot{\kappa} + A_1\ddot{\lambda} &= \mathfrak{F}_\phi \\ C\ddot{\psi} &= \mathfrak{F}_\psi \\ A_1\ddot{\phi} + A_2\ddot{\kappa} + A_3\ddot{\lambda} &= \mathfrak{F}_\kappa \\ A_1\ddot{\phi} + A_3\ddot{\kappa} + A_2\ddot{\lambda} &= \mathfrak{F}_\lambda \end{aligned}\right\} \quad (22)$$

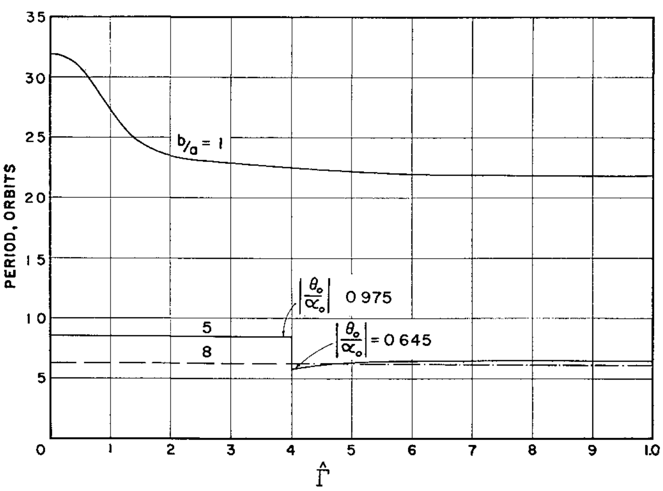


Fig 6 Period of least-damped antisymmetric longitudinal mode

where

$$\left.\begin{aligned} A &= \text{total moment of inertia about the } \xi_b \text{ axis} \\ A_1 &= A' + m'b(a + b) = B_1 \\ A_2 &= A' + m'b^2[1 - (m'/m)] = B_2 \\ A_3 &= (m'/m)(m'b^2) = B_3 \\ C &= \text{total moment of inertia about } \zeta_b \text{ axis} \end{aligned}\right\} \quad (23)$$

The computation of the generalized forces uses the expressions (17) and (18) together with the hinge friction couples $-c_1\dot{\alpha}$, $-c_1\dot{\beta}$ in the longitudinal case, and $-c_2\dot{\kappa}$, $-c_2\dot{\lambda}$ in the lateral case Again, the details may be found in Refs 4 and 7 The results are as follows:

Longitudinal

$$\left.\begin{aligned} \mathfrak{F}_\theta &= -\omega_0^2[2eB\sin\gamma + 3(A - C_b)\theta_b + 3B_1\alpha + 3B_1\beta] \\ \mathfrak{F}_\alpha &= -\omega_0^2[3B_1\theta_b + 3B_1\alpha + 2eB_1\sin\gamma + \Gamma_1(d\alpha/d\gamma)] \\ \mathfrak{F}_\beta &= -\omega_0^2[3B_1\theta_b + 3B_1\beta + 2eB_1\sin\gamma + \Gamma_1(d\beta/d\gamma)] \end{aligned}\right\} \quad (24)$$

Lateral

$$\left.\begin{aligned} \mathfrak{F}_\phi &= -\omega_0^2[F\phi - E(d\psi/d\gamma) + 4A_1\kappa + 4A_1\lambda] \\ \mathfrak{F}_\psi &= -\omega_0^2[E(d\phi/d\gamma) + (C - E)\psi] \\ \mathfrak{F}_\kappa &= -\omega_0^2[4A_1\phi + \Gamma_2(d\kappa/d\gamma) + (3A_1 + A_2)\kappa + A_3\lambda] \\ \mathfrak{F}_\lambda &= -\omega_0^2[4A_1\phi + A_3\kappa + \Gamma_2(d\lambda/d\gamma) + (3A_1 + A_2)\lambda] \end{aligned}\right\} \quad (25)$$

where

$$E = A_b - B_b + C_b$$
$$F = 4(A - E)$$

By combining Eqs (20) with (24) and (22) with (25), the equations of motion are obtained as follows:

Longitudinal

$$\begin{bmatrix} (Bd^2 + 3B_4) & B_1(d^2 + 3) & B_1(d^2 + 3) \\ B_1(d^2 + 3) & (B_2d^2 + \Gamma_1 d + 3B_1) & B_3d^2 \\ B_1(d^2 + 3) & B_3d^2 & (B_2d^2 + \Gamma_1 d + 3B_1) \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2eB\sin\gamma \\ -2eB_1\sin\gamma \\ -2eB_1\sin\gamma \end{bmatrix} \quad (26)$$

where $d = d/d\gamma$ and $B_4 = A - C_b$

Lateral

$$\begin{bmatrix} (Ad^2 + F) & -Ed & (A_1d^2 + 4A_1) & (A_1d^2 + 4A_1) \\ Ed & Cd^2 + (C - E) & 0 & 0 \\ (A_1d^2 + 4A_1) & 0 & (A_2d^2 + \Gamma_2 d + 3A_1 + A_2) & (A_3d^2 + A_3) \\ (A_1d^2 + 4A_1) & 0 & (A_3d^2 + A_3) & (A_2d^2 + \Gamma_2 d + 3A_1 + A_2) \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \kappa \\ \lambda \end{bmatrix} = 0 \quad (27)$$

As a result of the approximation of the eccentricity terms previously described, the ellipticity of the orbit appears only in the inhomogeneous "forcing" terms on the right-hand side of the longitudinal equations The left-hand side of these equations defines a homogeneous system of the sixth degree, and the lateral equations are a system of the eighth degree

Numerical Solutions

Numerical solutions were obtained to the foregoing equations of motion using the IBM 7090 digital computer at the University of Toronto Institute of Computer Scien e The

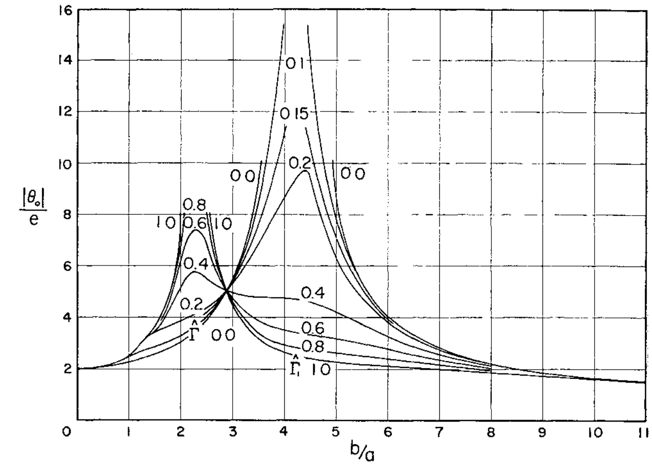


Fig 7 Forced oscillation in pitch caused by eccentricity of orbit

data used were as follows: for the longitudinal case,

$$m'/m_b = 0\ 005(b/a) \qquad A' = B' = \tfrac{1}{3}m'b^2$$
$$A_b = B_b = C_b = 0\ 6m_ba^2$$

and for the lateral case,

$$A_b = A_{b0} + I \qquad B_b = B_{b0}$$
$$C_b = C_{b0} + I \qquad A_{b0} = B_{b0} = C_{b0} = 0\ 4\ m_b\ a^2$$

where I is the moment of inertia of the roll-yaw coupler rod on the η_b axis (The difference between the satellite body radii of gyration in the longitudinal and lateral cases is not significant It results from the fact that the two cases were done somewhat independently) In addition, fixed yaw stabilizers of length $2b'$ and mass per unit length equal to that of the pitch-roll stabilizers were incorporated in the lateral solutions as an additional parameter Both sets of equations, (26) and (27), were solved for the characteristic modes The principal properties, period, and time to damp to half-amplitude were computed for a variety of values of stabilizer lengths and hinge damping parameters Equation (26) was also solved for the steady-state forced oscillations associated with the e terms on the right-hand side Representative results are shown in Figs 4–10 Both oscillatory and nonoscillatory characteristic modes were obtained, depending on the values of the parameters Figure 5 shows that damping of the body pitching motion to half-amplitude occurs in as little as $\frac{1}{3}$ orbit when $\hat{\Gamma}_1 = 0\ 4$ and $b/a = 5\ 0$ (Since the symmetric mode does not entail rotation of the satellite body, it is technically less significant than the antisymmetric mode) This result may be compared with the optimum value of $t_{1\ 2} = 0\ 095$ obtained by Zajac[9] for a pair of crossed dumbbells With the damping time of $\frac{1}{3}$ orbit, a transient

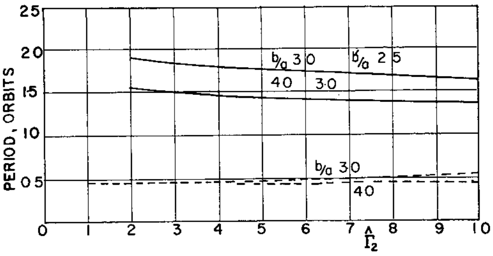


Fig 8 Period of least-damped lateral modes Solid line indicates antisymmetric modes, and dashed line indicates symmetric modes

would damp to $\frac{1}{8}$ of its initial amplitude in one orbit Figure 6 shows the periods of the modes of Fig 5 The jump at $\hat{\Gamma}_1 = 0\ 4$ occurs because there are two periodic antisymmetric modes, and theii dampings cross over at this point The ratios $|\theta_0/\alpha_0|$ shown on the figure are the modal amplitude ratios, which identify the "shape" of the mode, i e , the amplitude of the body pitch is 64 5% of the stabilizer pitch amplitude when $|\theta_0/\alpha_0| = 0\ 645$ Figuie 7 shows the steady-state forced oscillations of the body in pitch which are produced by ellipticity of the orbit Resonant behavioi occurs when the orbital period coincides with one of the characteristic periods It is clear that, in choosing design parameters such as b/a and $\hat{\Gamma}_1$, account must be taken of the expected eccentricity of the orbit and the required pointing accuracy The best peiformance is about $\theta_0 = 2e$ For example, if the difference between apogee and perigee is 100 miles for a low orbit, $e \doteq 0\ 012$ and $\theta_0 = 0\ 024$ rad $= 1\ 37°$ Design compromises might, however, require choices of $\hat{\Gamma}_1$ and b/a which would lead to values somewhat larger than this

Figuies 8 and 9 show iepresentative results for the period and damping of the lateial modes obtained The mode shapes are either symmetric or antisymmetric and look much the same when viewed in the yz plane as the longitudinal modes of Fig 4, except that the antisymmetric modes also include a rotation ψ of the body about the z axis For the cases shown, the period is about $1\frac{1}{2}$ to 2 oibits, and the damping time is rather long, the minimum value shown being about 1 35 orbits to half-amplitude when $b/a = 3\ 0$ and $b'/a = 2\ 5$ A word of explanation is necessary as to why there is damping of the yawing motion at all The explanation lies in the roll-yaw coupling, provided by the terms $-Ed\dot{\psi}$ and $Ed\dot{\phi}$ in the first and second of Eqs (27) The origin of these teims can be seen by considering the Coiiolis forces acting on elements of mass lying on the η_b axis when they have motion due to $\dot{\phi}$ or $\dot{\psi}$ It is precisely to increase this coupling that the added mass on the η_b axis, the roll-yaw coupler having moment of inertia I, was incorporated Thus any yawing motion of the body ($\dot{\psi}$) induces a rolling moment ($E\dot{\psi}$) which, in turn, sets up iolling motion of the body and of the roll stabilizers The relative rotations κ and λ which result then produce eneigy dissipation in the hinges, and the entire motion decays The effectiveness of the added roll-yaw coupler is shown in Fig 10, where the period and damping of the least-damped lateral mode are plotted as functions of the inertia of the roll-yaw coupler $t_{1/2}$ is reduced by it from 1 35 to less than 0 5 for $b/a = 3\ 0$

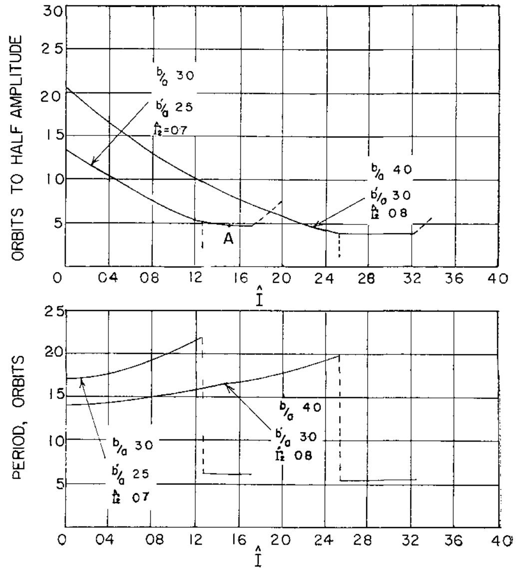


Fig 10 Effect of roll-yaw coupler on damping of least-damped lateral mode

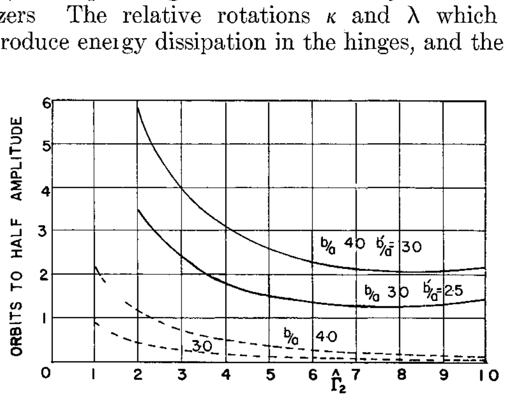


Fig 9 Damping of least-damped lateral modes Solid line indicates antisymmetric modes, and dashed line indicates symmetric modes

Optimization

It was of interest to search for the best damping that could be obtained by varying the four design parameters b/a, b'/a, $\hat{\Gamma}_2$, and $\hat{I}$, in order to establish the performance potential of the system

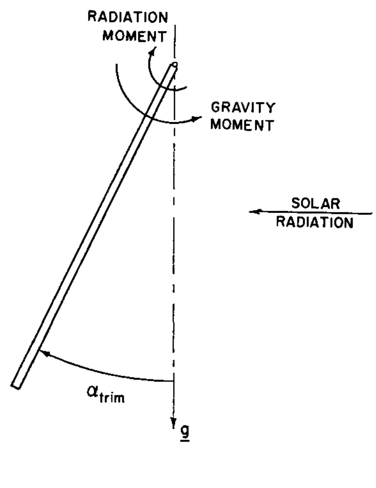
This was accomplished by a "method of steepest descent" analysis, in which the four parameters were varied stepwise in such a way as to increase the damping of the least-damped mode at each step The starting point was at A of Fig 10, and the result obtained after 12 iterations was as follows: orbits to $\frac{1}{2}$ amp $= 0\ 28$, $\hat{\Gamma}_2 = 0\ 637$, $b/a = 3\ 32$, $b'/a = 3\ 09$, and $\hat{I} = 0\ 218$

It is therefore evident that very rapid damping of both the lateral and longitudinal modes is possible when they are considered independently In an actual design situation, when both must be consideied jointly, some compromises would likely have to be made which would lead to slower damping than the best values just quoted

Effect of Solar Radiation Pressure

Solar radiation pressure may supply important disturbing foices, depending on the altitude, the orbit, and the particular

Fig 11 Equilibrium of solar radiation and gravity moments



design of the satellite Generally speaking, it pıovides periodic inputs of orbital period to both the longitudinal and the lateral systems If the vehicle is symmetric and the radiation is both weak and uniform, then it can be shown that the effect on the motion is second order In this context "weak" means that the "trim angle' $\alpha_{t\ im}$ of a stabilizer rod under steady conditions of gravity gradient and radiation pressure is small (see Fig 11)

The radiation incident on the vehicle will be uniform except for such effects as shadows cast on the rods by the body, and except when the vehicle passes into and out of the earth's shadow The moments associated with the latter effect have been calculated in Ref 4 and shown to produce motions that are small compared to those caused by orbit ellipticity for low-altitude satellites Now the radiation pressure is essentially independent of altitude, but the gravity moment decreases as ω^2, i e , as R^{-3} Hence, even though radiation is not important for a symmetric satellite at low altitudes, it does eventually become so as height increases Calculations for a typical rod have shown that the trim angle is less than 20° for altitudes less than about 4000 miles, but that it may become as large as 80° at synchıonous altitude Large trim angles imply that large-amplitude motions of the rods may be induced by the periodic action of the solar pressure and that the consequent lack of symmetry of the configuration may result in appreciable solar pressure torques, which may in turn cause significant body rotations Hence we may conclude that, for any given design of the kind just described, there is an effective "ceiling," above which the pointing accuıacy deteriorates to unacceptable values Since the cause of the eırors is the joint action of configuration asymmetry and solar pressuıe, it may be anticipated that reducing the former will be beneficial Thus, incorporating springs in the hinges so as to ıeduce the trim angles (i e , the asymmetry) is expected to result in increased ceilings However, no computed results are yet available to substantiate this expectation

References

[1] Roberson, R E , "Attitude control of satellites and space vehicles," *Advances in Space Science* (Academic Press, New York, 1960) Vol 2

[2] Etkin, B , *Dynamics of Flight* (John Wiley & Sons, Inc , New York, 1959), Eq (4 8, 6)

[3] Synge, J L and Griffith, B A , *Principles of Mechanics* (McGraw Hill Book Co , Inc , New York, 1942), Sec 6 5

[4] Etkin, B , "Attitude stability of articulated gravity-oriented satellites Part 1—General theory and motion in orbital plane," Univ of Toronto, Institute of Aerophysics Rept 89 (1962)

[5] Halfman, R L , *Dynamics* (Addison-Wesley Publishing Co Inc , Reading, Mass , 1962), Vol 1, Sec 5-3

[6] Knollman, G C and Pyron, B O , "Relative trajectories of objects ejected from a neaı satellite," AIAA J **1**, 424–429 (1963)

[7] Maeda, H , "Attitude stability of articulated gravity oriented satellites Part 2—Lateral motion," Univ of Toronto, Institute of Aerophysics Rept 93 (1963)

[8] MacNaughton, J D , ' Unfurlable metal structures for spacecraft," Can Aeronaut Space Inst J **9**, 103–116 (April 1963)

[9] Zajac, E E , "Damping of a gravitationally oriented two-body satellite," ARS J **32**, 1871–1875 (1962)

[10] Tinling, B E and Merrick, V K , The exploitation of inertial coupling in passive gravity-gradient stabilized satellites," AIAA Paper 63-342 (August 1963)

[11] Fletcher, H J , Rongved, L , and Yu, E Y , "Dynamics analysis of a two-body gravitationally oriented satellite ' Bell System Tech J **42**, 5 (September 1963)